

A Note on Curvature Fluctuation of Noncommutative Inflation

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Abstract

An elegant approach, which incorporates the effect of the stringy spacetime uncertainty relation, to calculate power spectra of fluctuations during inflation has been suggested by Brandenberger and Ho. In this approach, one of important features is the appearance of an upper bound on the comoving momentum k , at which the stringy spacetime uncertainty relation is saturated. As a result, the time-dependent upper bound leads us to choose naturally a set of initial vacua, in which the stringy uncertainty relation is saturated. In this note, with that set of vacua we calculate power spectrum of curvature fluctuation for a power law inflation, up to the leading order of a parameter describing the spacetime noncommutativity. It turns out that this choice of initial vacuum has a significant effect on the power spectrum of fluctuations.

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1 Introduction

String/M theory as a fundamental theory should produce a cosmology model describing our universe. However, our understanding so far to the string/M theory is quite incomplete, even we have not yet had a successful perturbative string theory in a time-dependent background. Although so, it is generically believed that some remarkable features, for instance, spacetime noncommutativity, of present string/M theory (quantum gravity theory) will manifest around the string scale in an effective field theory description.

On the other hand, according to the inflation model [1, 2, 3], the large scale structure of the universe we observed today is formed starting from quantum fluctuations during inflation; those tiny quantum fluctuations are stretched to cosmo scales with the expansion of the universe. With the precisely astronomical observations, now it becomes possible to observe the effects of spacetime structure around the string scale in the modern cosmology. Indeed, in recent years some authors studied the effect of spacetime noncommutativity on inflation models [4]-[16].

In string theory, there is a universal spacetime uncertainty relation, $\Delta t_p \Delta x_p \geq l_s$, where t_p and x_p are physical time and distance, and $l_s = M_s^{-1}$ is the string length scale. This stringy spacetime uncertainty relation (SSUR) holds not only for perturbative string theory, but also for nonperturbative string theory [17]. The SSUR implies that at the string scale spacetime is noncommutative, and field theory in such a background must be nonlocal. However, it turns out that it is not so easy to incorporate the spacetime noncommutativity to the effective description of field theory in inflation model. In a recent paper [10] by Brandenberger and Ho, an elegant approach is proposed, in which the spacetime noncommutativity is incorporated through replacing the usual product by star product of noncommutative field theory in the effective action describing the fluctuations of inflaton and metric. The effect of spacetime noncommutativity on the power spectrum of fluctuations with the Brandenberger-Ho's approach has been investigated in some recent papers. In particular, papers [11, 12, 13, 14] used the effect to fit the WMAP data [18], showing that the noncommutativity effect could produce a large running of spectrum index in some inflation models, but it is not enough to generate the suppression of angle power spectrum at large scales, as the WMAP data indicate. Myung et al [15] calculated the power spectrum of curvature fluctuation up to the second order of slow roll parameters and noncommutativity parameter. Following [10], Liu and Li in [16] discussed the scalar fluctuations of tachyon inflation in a noncommutative spacetime.

Since the background (the Friedmann-Robertson-Walker spacetime) is homogeneous and isotropic, in the Brandenberger-Ho's approach, the spacetime noncommutativity has

not any effect on the background evolution. However, two new features occur in this approach, compared to inflation model in the commutative (usual) spacetime. One of them is that the coupling between the fluctuation modes and the background becomes nonlocal in time. The other is that there exists a critical time for each mode, at which the SSUR is saturated, which indicates that the critical time is the one when the mode is generated, before that the mode does not exist. This critical time leads to a natural choice of initial vacuum for given a mode. However, when calculating the power spectrum of fluctuations, most of existing literature still choose the adiabatic vacuum as the initial vacuum. In this note we emphasize the significance of the natural choice of initial vacuum, and calculate the power spectrum of curvature fluctuation for a power law inflation in a noncommutative spacetime. It turns out that indeed this choice of vacuum has a significant effect on the power spectrum of fluctuations.

2 Curvature perturbation of noncommutative inflation

The model incorporating the SSUR can be written down as [10]

$$S = V \int_{k < k_0} d\tilde{\eta} d^3k z_k^2(\tilde{\eta}) (\phi'_{-k} \phi'_k - k^2 \phi_{-k} \phi_k), \quad (2.1)$$

where V denotes the total spatial volume and z_k is some smeared version of $z = a\dot{\phi}_0/H$ or the scale factor a over a range of time of characteristic scale $\Delta\tau = l_s^2 k$ with the overdot being the derivative with respect to the cosmic time, ϕ_k is the Fourier mode with comoving wave number k of scalar perturbation of metric, ϕ_0 represents the background inflaton, and H is the Hubble parameter (for more details, see also the appendix in [10]). In addition,

$$z_k^2 = z^2 y_k^2(\tilde{\eta}), \quad \tilde{z}_k(\tilde{\eta}) = a_{\text{eff}}(\tilde{\eta}). \quad (2.2)$$

Here the modified conformal time $\tilde{\eta}$ is defined via $d\tilde{\eta} = \tilde{z}_k^{-2} d\tau$, the coordinate time τ is related to the cosmic time t via $d\tau = a dt$, and the conformal time η is defined as usual as $d\eta = a^{-1} dt$. Namely, a flat Friedmann-Robertson-Walker metric can be written in the following forms in terms of different times

$$\begin{aligned} ds^2 &= dt^2 - a^2(t)(dr^2 + r^2 d\Omega^2), \\ &= a^2(\eta)(d\eta^2 - (dr^2 + r^2 d\Omega^2)), \\ &= a^{-2}(\tau)d\tau^2 - a^2(\tau)(dr^2 + r^2 d\Omega^2). \end{aligned} \quad (2.3)$$

Finally, y_k^2 and a_{eff} in (2.2) are given by

$$y_k = (\beta_k^- \beta_k^+)^{1/4}, \quad a_{\text{eff}}^2 = (\beta_k^+ / \beta_k^-)^{1/2}, \quad (2.4)$$

where

$$\beta^\pm(\tau) = \frac{1}{2}(a^{\pm 2}(\tau - l_s^2 k) + a^{\pm 2}(\tau + l_s^2 k)). \quad (2.5)$$

Note that there exists an upper bound k_0 on the wave number in the effective action (2.1), which is given by

$$k \leq k_0(\tau) = a_{\text{eff}}(\tau)/l_s. \quad (2.6)$$

This has an important consequence on the perturbation. It implies that given a wave number there is a critical time $\tau_0 = \tau_0(k)$. At the time when the equation (2.6) holds, the SSUR is saturated. This indicates that before the critical time the mode k should not exist in order the SSUR to hold.

The equation of scalar perturbation can be written as

$$\mu_k'' + (k^2 - \frac{z_k''}{z_k})\mu_k = 0, \quad (2.7)$$

where a prime denotes the derivative with respect to $\tilde{\eta}$ and $\mu_k = z_k \phi_k$. Defining the slow-roll parameters

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2} \left(\frac{\dot{\phi}_0}{H} \right)^2, \quad \delta = \frac{\ddot{\phi}_0}{H \dot{\phi}_0}, \quad (2.8)$$

and introducing a new parameter describing the noncommutativity of spacetime [14]

$$\mu = \left(\frac{kH}{aM_s^2} \right)^2, \quad (2.9)$$

one has

$$\begin{aligned} \frac{z_k''}{z_k} &= 2(aH)^2 \left(1 + \frac{5}{2}\epsilon - \frac{3}{2}\delta - 2\mu \right), \\ aH &= -\frac{1}{\tilde{\eta}}(1 + \epsilon + \mu), \end{aligned} \quad (2.10)$$

up to the leading order of ϵ , δ and μ . The equation (2.7) is then changed to

$$\mu_k'' + (k^2 - \frac{\nu^2 - 1/4}{\tilde{\eta}^2})\mu_k = 0, \quad (2.11)$$

where $\nu = 3/2 + 2\epsilon + \delta$. It is interesting to note that the equation (2.11) has a same form as the one for the usual case without spacetime noncommutativity and that the only difference is to replace the conformal time η by the modified one $\tilde{\eta}$. Therefore once

obtaining the solution of the equation (2.11) and choosing an appropriate initial vacuum $|0\rangle$, one has the power spectrum of the curvature fluctuation $\mathcal{R} = \mu_k/z_k$,

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} \frac{\langle 0 | \hat{\mu}_k^+ \hat{\mu}_k | 0 \rangle}{z_k^2}. \quad (2.12)$$

It is well-known that there does not exist an unique vacuum for quantum fields in curved spaces. As a result, the key point is to choose an appropriate vacuum, which could describe correctly physics happened in the situation under consideration. Usually when one deals with quantum fluctuations during inflation, the adiabatic vacuum is chosen. However, it is not suitable to choose the adiabatic vacuum here since for a given mode k , there exists a critical time τ_0 given by equation (2.6), before that the mode does not exist.

To choose an appropriate initial vacuum for the present situation, let us recall the approach developed by Danielsson [19] (for earlier discussion see [20]), which is used to discuss the effect of the transplanckian physics on inflation [19, 21]. There is also an upper bound on the wave number, beyond which the effective field theory description invalidates. Therefore the situation for the transplanckian physics is quite similar to the present case. In order to incorporate the effect of the upper bound (2.6), we can therefore follow the approach proposed by Danielsson [19].

The momentum which is conjugate to μ_k is

$$\pi_k = \mu'_k - \frac{z'_k}{z_k} \mu_k. \quad (2.13)$$

Since the vacuum will be dependent on time, the Heisenberg picture is the most convenient one to quantize the system, in which operators evolve with time, but states do not. In terms of time dependent oscillators, we have

$$\begin{aligned} \hat{\mu}_k(\tilde{\eta}) &= \frac{1}{\sqrt{2k}} (\hat{a}_k(\tilde{\eta}) + \hat{a}_{-k}^+(\tilde{\eta})), \\ \hat{\pi}_k(\tilde{\eta}) &= -i\sqrt{\frac{k}{2}} (\hat{a}_k(\tilde{\eta}) - \hat{a}_{-k}^+(\tilde{\eta})). \end{aligned} \quad (2.14)$$

The oscillators can be expressed in terms of their values at a time $\tilde{\eta}_0 = \tilde{\eta}_0(\tau_0)$ through a Bogoliubov transformation

$$\begin{aligned} \hat{a}_k(\tilde{\eta}) &= u_k(\tilde{\eta}) \hat{a}_k(\tilde{\eta}_0) + v_k(\tilde{\eta}) \hat{a}_{-k}^+(\tilde{\eta}_0), \\ \hat{a}_{-k}^+(\tilde{\eta}) &= u_k^*(\tilde{\eta}) \hat{a}_{-k}^+(\tilde{\eta}_0) + v_k^*(\tilde{\eta}) \hat{a}_k(\tilde{\eta}_0). \end{aligned} \quad (2.15)$$

Substituting this into (2.14) yields

$$\begin{aligned} \hat{\mu}_k(\tilde{\eta}) &= f_k(\tilde{\eta}) \hat{a}_k(\tilde{\eta}_0) + f_k^*(\tilde{\eta}) \hat{a}_{-k}^+(\tilde{\eta}_0), \\ i\hat{\pi}_k(\tilde{\eta}) &= g_k(\tilde{\eta}) \hat{a}_k(\tilde{\eta}_0) - g_k^*(\tilde{\eta}) \hat{a}_{-k}^+(\tilde{\eta}_0), \end{aligned} \quad (2.16)$$

where

$$f_k(\tilde{\eta}) = \frac{1}{\sqrt{2k}}(u_k(\tilde{\eta}) + v_k^*(\tilde{\eta})), \quad g_k(\tilde{\eta}) = \sqrt{\frac{k}{2}}(u_k(\tilde{\eta}) - v_k^*(\tilde{\eta})). \quad (2.17)$$

Here $f_k(\tilde{\eta})$ is a solution of equation (2.7), and the normalization condition is

$$|u_k|^2 - |v_k|^2 = 1. \quad (2.18)$$

Obviously a reasonable choice for the initial vacuum is

$$\hat{a}_k(\tilde{\eta}_0)|0, \tilde{\eta}_0\rangle = 0, \quad (2.19)$$

since the mode k does not exist when $\tilde{\eta} < \tilde{\eta}_0$ (note that here $\tilde{\eta}$ is always negative). In this case, one can see from the relation (2.14) that the field and its conjugate momentum has a simple relation

$$\hat{\pi}_k(\tilde{\eta}_0)|0, \tilde{\eta}_0\rangle = ik\hat{\mu}_k(\tilde{\eta}_0)|0, \tilde{\eta}_0\rangle. \quad (2.20)$$

This choice of vacuum has a simple physical interpretation [19]. It corresponds to a state which minimizes the uncertainty at the time $\tilde{\eta} = \tilde{\eta}_0$. This is consistent with the statement mentioned above that at the time $\tilde{\eta}_0$ the SSUR is saturated. Also it in turn justifies the choice (2.19) of initial vacuum.

For the equation (2.14), one has the following solution

$$f_k = A_k\sqrt{-\tilde{\eta}}J_\nu(-k\tilde{\eta}) + B_k\sqrt{-\tilde{\eta}}Y_\nu(-k\tilde{\eta}), \quad (2.21)$$

where J_ν and Y_ν are Bessel functions of the first and second kind, respectively, and A_k and B_k are two complex constants.

Now we consider a power-law inflation with a scale factor $a = a_0 t^p$, which could be produced by a scalar field with an exponential potential $V = V_0 \exp(-\sqrt{2/p}\phi_0/M_p)$. In this case, $\dot{\phi}_0 = \sqrt{2p}M_p/t$, $H = p/t$, and $z = a\sqrt{2/p}M_p$. Here M_p denotes the induced Planck mass. In the slow roll approximation, for the power law inflation, one has $\epsilon = -\delta = 1/p$, so $\nu = 3/2 + 1/p$. In fact, for the power law inflation, the slow roll approximation is not necessary, the exact value of ν is $3/2 + 1/(p-1)$. It is seen that the slow roll approximation is quite well if $p \gg 1$. Therefore, in what follows we will take $\nu = 3/2 + 1/(p-1)$. For the solution (2.21) we have

$$g_k = -ik\sqrt{-\tilde{\eta}}(A_k J_{\nu-1}(-k\tilde{\eta}) + B_k Y_{\nu-1}(-k\tilde{\eta})). \quad (2.22)$$

Using (2.16) we obtain

$$\begin{aligned} u_k &= \sqrt{-k\tilde{\eta}/2}[A_k J_\nu(-k\tilde{\eta}) + B_k Y_\nu(-k\tilde{\eta}) - i(A_k J_{\nu-1}(-k\tilde{\eta}) + B_k Y_{\nu-1}(-k\tilde{\eta}))], \\ v_k^* &= \sqrt{-k\tilde{\eta}/2}[A_k J_\nu(-k\tilde{\eta}) + B_k Y_\nu(-k\tilde{\eta}) + i(A_k J_{\nu-1}(-k\tilde{\eta}) + B_k Y_{\nu-1}(-k\tilde{\eta}))]. \end{aligned} \quad (2.23)$$

The normalization condition (2.18) yields

$$A_k B_k^* - A_k^* B_k = -i \frac{\pi}{2}. \quad (2.24)$$

The vacuum choice (2.19) together with the Bogoliubov transformation (2.15) implies that $v_k^*(\tilde{\eta}_0) = 0$. From (2.23) we then have

$$A_k = -\frac{Y_\nu(-k\tilde{\eta}_0) + iY_{\nu-1}(-k\tilde{\eta}_0)}{J_\nu(-k\tilde{\eta}_0) + iJ_{\nu-1}(-k\tilde{\eta}_0)} B_k. \quad (2.25)$$

Equations (2.24) and (2.25) determine those two constants

$$\begin{aligned} |A_k|^2 &= -\frac{\pi^2}{8} k \tilde{\eta}_0 (Y_\nu^2(-k\tilde{\eta}_0) + Y_{\nu-1}^2(-k\tilde{\eta}_0)), \\ |B_k|^2 &= -\frac{\pi^2}{8} k \tilde{\eta}_0 (J_\nu^2(-k\tilde{\eta}_0) + J_{\nu-1}^2(-k\tilde{\eta}_0)), \\ A_k B_k^* &= -i \frac{\pi}{4} + \frac{\pi^2}{8} k \tilde{\eta}_0 (J_\nu(-k\tilde{\eta}_0) Y_\nu(-k\tilde{\eta}_0) + J_{\nu-1}(-k\tilde{\eta}_0) Y_{\nu-1}(-k\tilde{\eta}_0)), \\ A_k^* B_k &= i \frac{\pi}{4} + \frac{\pi^2}{8} k \tilde{\eta}_0 (J_\nu(-k\tilde{\eta}_0) Y_\nu(-k\tilde{\eta}_0) + J_{\nu-1}(-k\tilde{\eta}_0) Y_{\nu-1}(-k\tilde{\eta}_0)). \end{aligned} \quad (2.26)$$

Substituting these into (2.12), we obtain the exact power spectrum of the curvature fluctuation

$$\begin{aligned} \mathcal{P}_{\mathcal{R}}(k) &= \frac{k^4 \tilde{\eta}_0 \tilde{\eta}}{16 z^2 y_k^2(\tilde{\eta})} [(Y_\nu^2(-k\tilde{\eta}_0) + Y_{\nu-1}^2(-k\tilde{\eta}_0)) J_\nu^2(-k\tilde{\eta}) \\ &\quad + (J_\nu^2(-k\tilde{\eta}_0) + J_{\nu-1}^2(-k\tilde{\eta}_0)) Y_\nu^2(-k\tilde{\eta}) \\ &\quad - 2(J_\nu(-k\tilde{\eta}_0) Y_\nu(-k\tilde{\eta}_0) + J_{\nu-1}(-k\tilde{\eta}_0) Y_{\nu-1}(-k\tilde{\eta}_0)) J_\nu(-k\tilde{\eta}) Y_\nu(-k\tilde{\eta})] \end{aligned} \quad (2.27)$$

If one does not consider the effect of spacetime noncommutativity, replacing $\tilde{\eta} \rightarrow \eta$ and setting $y_k = 1$, one can then obtain from (2.27) the spectrum of curvature perturbation for a power-law inflation taking into account the transplanckian physics [21]. Note that the authors of [21] only give the spectrum for a massless scalar field for a power law inflation. Therefore compared to the case of commutative spacetime, due to the spacetime noncommutativity, two new features appear in (2.27): (1) the modified conformal time $\tilde{\eta}$ replaces the usual conformal time η ; (2) a new factor y_k^2 occurs in (2.27). if considering $\tilde{\eta}_0 \rightarrow -\infty$ and $\tilde{\eta} \rightarrow 0^-$, one is then led to the spectrum obtained in the adiabatic vacuum.

Strictly speaking, the power spectrum should be calculated when the mode k crosses the Hubble horizon. That is,

$$k^2 = z_k''/z_k. \quad (2.28)$$

From (2.11), we can see that it corresponds to $\tilde{\eta} = -\sqrt{\nu^2 - 1/4}/k$. The initial time $\tilde{\eta}_0$ is related to the coordinate time τ_0 via $\tilde{\eta}_0 = \tilde{\eta}_0(\tau_0)$, while the latter is given via the equation (2.6). For a power law inflation, it is [10]

$$\tau_0 = \left(\left(\frac{kl_s}{\alpha_0} \right)^{2(p+1)/p} + k^2 l_s^4 \right)^{1/2}, \quad (2.29)$$

where $\alpha_0 = ((p+1)^p a_0)^{1/(p+1)}$. The scale factor has the form $a = \alpha_0 \tau^{p/(p+1)}$ in terms of the time τ . Further, the modified conformal time $\tilde{\eta}$ has the relation to the coordinate time τ ,

$$\tilde{\eta} = - \left(1 + \frac{p+1}{p} \mu \right) \frac{p+1}{p-1} \frac{1}{\alpha_0^2 \tau^{(p-1)/(p+1)}}, \quad (2.30)$$

where

$$\mu = \frac{p^2}{(p+1)^2} \left(\frac{kl_s^2}{\tau} \right)^2. \quad (2.31)$$

Therefore the power spectrum of curvature fluctuation is

$$\begin{aligned} \mathcal{P}_{\mathcal{R}}(k) &= \frac{k^{3-2p/(p-1)} p}{32M_p^2 \alpha_0^6 \tau_0^{(p-1)/(p+1)}} \left(\alpha_0^2 \sqrt{\nu^2 - 1/4} \right)^{(3p-1)/(p-1)} \\ &\times \left(\frac{p-1}{p+1} \right)^{(p+1)/(p-1)} \left(1 - \frac{4p}{p-1} \mu_k \right) \left(1 + \frac{p+1}{p} \mu_0 \right) \\ &\times [Y_\nu^2(-k\tilde{\eta}_0) + Y_{\nu-1}^2(-k\tilde{\eta}_0)] J_\nu^2(\sqrt{\nu^2 - 1/4}) \\ &+ (J_\nu^2(-k\tilde{\eta}_0) + J_{\nu-1}^2(-k\tilde{\eta}_0)) Y_\nu^2(\sqrt{\nu^2 - 1/4}) \\ &- 2(J_\nu(-k\tilde{\eta}_0) Y_\nu(-k\tilde{\eta}_0) + J_{\nu-1}(-k\tilde{\eta}_0) Y_{\nu-1}(-k\tilde{\eta}_0)) \\ &\times J_\nu(\sqrt{\nu^2 - 1/4}) Y_\nu(\sqrt{\nu^2 - 1/4})]. \end{aligned} \quad (2.32)$$

Here

$$\mu_0 = \frac{p^2}{(p+1)^2} \left(\frac{kl_s^2}{\tau_0} \right)^2, \quad \mu_k = \frac{p^2}{(p+1)^2} \left(\frac{kl_s^2}{\tau_k} \right)^2, \quad (2.33)$$

and

$$\tau_k = \left(\frac{p+1}{p-1} \right)^{(p+1)/(p-1)} \left(\frac{k}{\alpha_0^2 \sqrt{\nu^2 - 1/4}} \right)^{(p+1)/(p-1)}. \quad (2.34)$$

Now we can discuss two limits, namely the UV region and IR region. In the UV region, the first term in (2.29) is dominant. In this case, the power spectrum can be approximately expressed as

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^{-\frac{2}{p-1} + \frac{1}{p}} p \alpha_0^{\frac{4}{p-1}}}{32M_p^2} \left(\frac{\alpha_0}{l_s} \right)^{\frac{p-1}{p}} \left(\nu^2 - \frac{1}{4} \right)^{\frac{3p-1}{2(p+1)}} \left(\frac{p-1}{p+1} \right)^{(p+1)/(p-1)}$$

$$\begin{aligned}
& \times \left(1 + \frac{\alpha_0^2 l_s^2}{2} \left(\frac{\alpha_0}{k l_s} \right)^{2/p} - \frac{4p}{p-1} \mu_k \right) \\
& \times [(Y_\nu^2(-k\tilde{\eta}_0) + Y_{\nu-1}^2(-k\tilde{\eta}_0)) J_\nu^2(\sqrt{\nu^2 - 1/4}) \\
& + (J_\nu^2(-k\tilde{\eta}_0) + J_{\nu-1}^2(-k\tilde{\eta}_0)) Y_\nu^2(\sqrt{\nu^2 - 1/4}) \\
& - 2(J_\nu(-k\tilde{\eta}_0) Y_\nu(-k\tilde{\eta}_0) + J_{\nu-1}(-k\tilde{\eta}_0) Y_{\nu-1}(-k\tilde{\eta}_0)) \\
& \times J_\nu(\sqrt{\nu^2 - 1/4}) Y_\nu(\sqrt{\nu^2 - 1/4})], \tag{2.35}
\end{aligned}$$

where

$$-k\tilde{\eta}_0 = \frac{p+1}{p-1} \frac{1}{\alpha_0 l_s} \left(\frac{k l_s}{\alpha_0} \right)^{1/p}. \tag{2.36}$$

In the adiabatic vacuum, which corresponds to the case $\tilde{\eta}_0 \rightarrow -\infty$, the spectrum has a form $\mathcal{P}_{\mathcal{R}}(k) \sim k^{-2/(p-1)}$ with spectrum index $n_s = 1 - 2/(p-1)$, up to the leading order. Now for the Danielsson vacuum (2.19), it becomes $\mathcal{P}_{\mathcal{R}}(k) \sim k^{-2/(p-1)+1/p} [(J_\nu(y) Y_\mu(x) - Y_\nu(y) J_\mu(x))^2 + (J_{\nu-1}(y) Y_\nu(x) - Y_{\nu-1}(y) J_\nu(x))^2]$, where $y = -k\tilde{\eta}_0$ and $x = \sqrt{\nu^2 - 1/4}$. In this case, the spectrum index will be larger than the one for the adiabatic vacuum.

On the other hand, in the IR region, the second term in (2.29) is dominant. In that case, the power spectrum is approximated to

$$\begin{aligned}
\mathcal{P}_{\mathcal{R}}(k) &= \frac{k^{\frac{2}{p+1}-\frac{2}{p-1}} p \alpha_0^{\frac{4}{p-1}}}{32 M_p^2 l_s^{2(p-1)/(p+1)}} \left(\nu^2 - \frac{1}{4} \right)^{\frac{3p-1}{2(p-1)}} \left(\frac{p-1}{p+1} \right)^{(p+1)/(p-1)} \\
& \times \left(1 + \frac{p}{p+1} - \frac{4p}{p-1} \mu_k \right) \\
& \times [(Y_\nu^2(-k\tilde{\eta}_0) + Y_{\nu-1}^2(-k\tilde{\eta}_0)) J_\nu^2(\sqrt{\nu^2 - 1/4}) \\
& + (J_\nu^2(-k\tilde{\eta}_0) + J_{\nu-1}^2(-k\tilde{\eta}_0)) Y_\nu^2(\sqrt{\nu^2 - 1/4}) \\
& - 2(J_\nu(-k\tilde{\eta}_0) Y_\nu(-k\tilde{\eta}_0) + J_{\nu-1}(-k\tilde{\eta}_0) Y_{\nu-1}(-k\tilde{\eta}_0)) \\
& \times J_\nu(\sqrt{\nu^2 - 1/4}) Y_\nu(\sqrt{\nu^2 - 1/4})], \tag{2.37}
\end{aligned}$$

where

$$-k\tilde{\eta}_0 = \frac{p+1}{p-1} \frac{(k l_s)^{2/(p+1)}}{\alpha_0^2 l_s^2}. \tag{2.38}$$

Here one can see that the spectrum index increases, compared to the case of the UV region. The Danielsson vacuum enhances the blue tilt.

An extremal IR limit is that $\tilde{\eta} \rightarrow \tilde{\eta}_0$. This implies that the mode is generated outside the Hubble horizon; once generated, it becomes classical fluctuation. In this case, one has from (2.27),

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^{4/(p+1)} p}{2^{(4p+2)/(p+1)} \pi^2 M_p^2 \alpha_0^4 l_s^{(6p-2)/(p+1)}} \left(1 - \frac{p}{p+1} \frac{5}{4 \alpha_0^2 l_s^2} \left(\frac{k l_s}{\alpha_0} \right)^{2/p} \right), \tag{2.39}$$

which is same as the one obtained in [13]. From (2.39), one can see that the spectrum index $n_s = 1 + 4/(p+1)$ and the spectrum becomes blue tilt. Note that for modes generated inside the horizon, the spectrum is always red tilt with $n_s = 1 - 2/(p-1)$ if without the spacetime noncommutativity. The noncommutativity leads a blue tilt spectrum. The Danielsson vacuum enhances the trend from the red tilt to blue tilt, which can be seen from (2.35) and (2.37). The comoving wave number when the spectrum is from red tilt to blue tilt can be determined by studying (2.32). This needs further analysis and numerical calculation, which are currently under investigation.

In addition, sometimes people calculate the spectrum at the end of inflation (super-horizon scale), namely, $\tilde{\eta} \rightarrow 0^-$. In this case, Y_ν dominates over J_ν . Up to the leading order, the spectrum can be simplified to

$$\begin{aligned} \mathcal{P}_{\mathcal{R}}(k) = & -\frac{k^{1-2/(p-1)} p \Gamma^2(\nu) \alpha_0^{2(p+1)/(p-1)} \tilde{\eta}_0}{2^{2-2/(p-1)} \pi^2 M_p^2} \left(\frac{p-1}{p+1} \right)^{2p/(p-1)} \\ & \times [J_\nu^2(-k\tilde{\eta}_0) + J_{\nu-1}^2(-k\tilde{\eta}_0)], \end{aligned} \quad (2.40)$$

where τ_0 is given by (2.29) and $-k\tilde{\eta}_0$ is given by (2.30). If one further takes limit $k\tilde{\eta}_0 \rightarrow -\infty$ and expands to leading order for small k , one has

$$\begin{aligned} \mathcal{P}_{\mathcal{R}}(k) = & \frac{k^{-2/(p-1)} p \Gamma^2[3/2 + 1/(p-1)] \alpha_0^{2(p+1)/(p-1)}}{2^{(p-3)/(p-1)} \pi^3 M_p^2} \left(\frac{p-1}{p+1} \right)^{2p/(p-1)} \\ & \times \left(1 + \frac{P_0}{H_0} \sin \left(\frac{2p}{p-1} \frac{P_0}{H_0} - \frac{\pi p}{p-1} \right) \right), \end{aligned} \quad (2.41)$$

where

$$P_0 = \frac{k}{\alpha_0 (k l_s^2)^{p/(p+1)}}, \quad H_0 = \frac{\alpha_0 p}{p+1} (k l_s^2)^{-1/(p+1)}, \quad (2.42)$$

they are physical momentum of mode k and the Hubble parameter at the time $\tau_0 = k l_s^2$, respectively. Clearly the factor in the second line of (2.41) is the effect of the “trans-planckian physics” [19, 21].

3 Conclusions

Studying spacetime structure around a fundamental scale (planck scale or string scale) is an interesting issue in its own right. The present form of string/M theory tells us that spacetime is noncommutative around the string scale. Naturally it is interesting to ask whether there are observable effects of the spacetime noncommutativity in experiments. The precision observations to the microwave background radiation of the universe make

possible to see the effect. On the other hand, observation data can in turn also give a constraint on the string scale. In the approach suggested by Brandenberger and Ho [10], the spacetime noncommutativity causes two new and interesting features in calculating the power spectrum of fluctuations during inflation. In this note we stressed the consequence of existence of a critical time for a given mode. The critical time leads to a natural choice of initial vacuum. Following the approach developed by Danielsson [19, 21], we calculated the power spectrum of curvature fluctuation for a power law inflation. It turns out that the choice of initial vacuum has a significant effect on the power spectrum. Compared to the results given in [11, 12, 13], which are obtained in the case of the adiabatic vacuum, from our results (2.35) and (2.37), it can be derived that the spectrum index increases and the running of spectrum index becomes larger, which is welcome by the WMAP data. For modes generated inside the Hubble horizon, the spectrum is red tilt, $n_s = 1 - 2/(p - 1)$ in the adiabatic vacuum (The spectrum is always red tilt if without the spacetime noncommutativity). In the extremal IR limit, where the mode is generated outside the Hubble horizon, the spectrum is blue tilt, $n_s = 1 + 4/(p + 1)$ [see (2.39)]. In the Danielsson vacuum, the spectrum index increases, compared to the adiabatic vacuum, it is favor to the flow of spectrum from red tilt to blue tilt. The differences of spectrum index and its running between the Danielsson vacuum and adiabatic vacuum have to be numerically determined by (2.32).

Acknowledgments

The author would like to thank Qing-Guo Huang, Miao Li and Bin Wang for useful discussions, and to express his gratitude to the Physics Department, Baylor University for its hospitality. This work was supported in part by a grant from Chinese Academy of Sciences, a grant No. 10325525 from NSFC, and by the Ministry of Science and Technology of China under grant No. TG1999075401.

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